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# SIMPLIFIED ROTATING TIRE MODELS BASED ON CYLINDRICAL SHELLS WITH FREE BOUNDARY CONDITIONS

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ABSTRACT

A number of studies have been previously concerned with vibrations of rotating tires. In order to capture detailed dynamics of a rotating tire, complex numerical models have been developed accounting for the non-homogeneous structural properties of the tire belt and sidewall, structural-acoustic coupling and complex geometry description (including that of the wheel rim), under varying rotation speeds. Although a remarkable amount of details can be successfully modelled using such an approach, it can be rather difficult to parameterize the model, i.e. to establish a clear-cut relationship between a geometrical or material property of a tire and the corresponding change in the tire dynamics. Thus an alternative approach is employed within the scope of this study by modelling simplified tire geometries. The focus of the study is put onto modelling vibrations of the tire belt, which could be approximated by a pressurised thin cylindrical shell having free boundaries. Then the mode shapes and resonance frequencies of the belt are calculated as functions of the material and geometrical properties of the shell and the speed of the shell rotation. The resonance frequencies and the mode shapes of the free rotating cylindrical shell could be used to form a foundation for the forced vibration response and thus offer a possibility to couple the free tire belt to models of the sidewall and the air cavity inside the tire using, for example, mobility-impedance based methods.

## INTRODUCTION

The effects of rotation significantly alter the dynamical behaviour of automotive tires. In non-rotating tires, vibrations are characterized by pairs of traveling waves that propagate in opposite directions but have the same speed of propagation. This results in formation of a standing wave (vibration mode). On the contrary, if the tire rotates, the two waves travel in opposite directions with different speeds. This is primarily due to the Coriolis effects which alter the propagation speeds of the two waves.

Although a remarkable amount of details can be successfully modelled using complex numerical tire models, it can be rather difficult to parameterize the model, i.e. to establish direct relationships between geometrical and material properties of a tire and the corresponding change in its dynamics. Thus an alternative approach is employed within the scope of this study by modelling simplified tire geometries. The focus of the study is put onto modelling vibrations of the tire belt, which could be approximated by a pressurised thin cylindrical shell.

The vibrations of shells have been attracting interest of many researches for over a century, including also vibrations of rotating shell-like structures. The mathematical description of vibrations of shell-like structures is perhaps not the simplest one and there have been many shell theories developed in the past trying to capture only the most important features that govern the structural response of vibrating shells [1]. Nevertheless, closed form solutions for free vibration problems of shell-like structures are possible only for particular geometries and in combinations with particularly convenient boundary conditions. If the effects of rotation are to be studied as well, then it becomes even more difficult to come up with closed form expressions for the natural frequencies. For example, Huang and Soedel [2] developed a set of equations of motion for a rotating shell and solved the free and forced vibration problem assuming simply supported boundary conditions. The natural frequencies were calculated as roots of a characteristic polynomial, which was shown by the authors to be bi-cubic only if the shell does not rotate. On the other hand, other types of boundary conditions have been considered as well, see for example the work of Warburton [3], who solved the free vibration problem based on Flügge shell equations with either both ends clamped or both ends free. This is a particularly involved procedure even though the shell did not rotate. Also the output of the model is the length of a shell resonating at a given frequency rather than the resonance frequency of the shell having a certain length. Nevertheless, the attractiveness of the method is that exact mode shapes and natural frequencies can be obtained.

In this paper the method is developed further to cover also for rotating thin cylindrical shells having free boundaries. However, instead of using Flügge’s equations, the equations of motion developed by Huang and Soedel [2] are used, and the free vibration problem is solved exactly for both ends of the shell free. This is thought to be a more useful set of boundary conditions if the shell is to be used for modelling a tire belt. This is due to an unnecessary constraint of the infinite radial stiffness of the sidewall if the simply supported boundary conditions are considered. The resonance frequencies and the mode shapes of the rotating shell could be used to form a basis for a forced vibration response of the shell and thus offer a possibility to couple the free tire belt to sidewall and the air cavity models using, for example, mobility impedance based methods. The mathematical model is developed in the first section, and an example shell is considered in the second section, which is followed by a conclusions section.

## MATHEMATICAL MODEL

The thin cylindrical shell, used to approximately model the tire belt, is shown schematically in Figure 1.

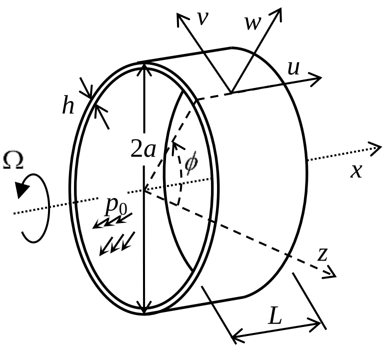


Figure 1: The rotating cylindrical shell

The equations of motions for used in this study are those originally developed by Huang and Soedel [2]. In case of free vibration they reduce to:

$$\left((1-\mu)K+2N_{\phi,i}\right)\frac{\partial^2 u}{\partial\phi^2}+2a\left(\frac{K(1+\mu)}{2}\frac{\partial^2 v}{\partial x\partial\phi}+K\left(a\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial w}{\partial x}\right)-\rho h a\frac{\partial^2 u}{\partial t^2}\right)=0 \tag{1}$$

$$\begin{aligned}&2D\left(\frac{\partial^3 w}{\partial\phi^3}+\frac{\partial^3 w}{\partial x^2\partial\phi}a^2\right)+a^2(\mu-1)\left(Ka^2+D\right)\frac{\partial^2 v}{\partial x^2}-2\left(\left(K+N_{\phi,i}\right)a^2+D\right)\frac{\partial^2 v}{\partial\phi^2}-\\&-a^2\left(aK(1+\mu)\frac{\partial^2 u}{\partial x\partial\phi}-2\rho h a^2\frac{\partial^2 v}{\partial t^2}+\left(4N_{\phi,i}+2K\right)\frac{\partial w}{\partial\phi}-4\rho h a^2\Omega\frac{\partial w}{\partial t}+2v\left(\rho h a^2\Omega^2-N_{\phi,i}\right)\right)=0\end{aligned} \tag{2}$$

$$\begin{aligned}&D\left(\frac{\partial^4 w}{\partial\phi^4}-\frac{\partial^3 v}{\partial\phi^3}-a^2\frac{\partial^3 v}{\partial x^2\partial\phi}+2a^2\frac{\partial^4 w}{\partial x^2\partial\phi^2}+a^4\frac{\partial^4 w}{\partial x^4}\right)+\\&+a^2\left(\mu K a\frac{\partial u}{\partial x}-N_{\phi,i}\frac{\partial^2 w}{\partial\phi^2}+\rho h a^2\frac{\partial^2 w}{\partial t^2}+\left(2N_{\phi,i}+K\right)\frac{\partial v}{\partial\phi}-2\rho h a^2\Omega\frac{\partial v}{\partial t}+\left(K+N_{\phi,i}-\rho h a^2\Omega^2\right)w\right)=0\end{aligned} \tag{3}$$

where:

| | | |
|---|---|---|
| $a$ | = | shell radius, |
| $h$ | = | shell thickness, |
| $L$ | = | shell length, |
| $p_0$ | = | inflation pressure, |
| $u$ | = | axial displacement, |
| $v$ | = | tangential displacement, |
| $w$ | = | radial displacement, |
| $x$ | = | axial coordinate, |

| | | |
|---|---|---|
| $\phi$ | = | tangential coordinate, |
| $z$ | = | radial coordinate, |
| $\mu$ | = | Poisson's ratio, |
| $E$ | = | Young's modulus, |
| $\rho$ | = | mass density, |
| $N_{\phi,i}$ | = | initial tension in the tangential direction, |
| $\Omega$ | = | rotation speed, |
| $D=\dfrac{Eh^3}{12\left(1-\mu^2\right)}$ | = | bending stiffness, and |
| $K=\dfrac{Eh}{1-\mu^2}$ | = | membrane stiffness. |

The initial tension in the tangential direction is given by:

$$N_{\phi,i}=\rho ha^2\Omega+ap_0 \tag{4}$$

where the first term is due to centrifugal forces and the second term is due to the initial inflation pressure of the shell. Substituting:

$$u=U_0 e^{\frac{\alpha x}{a}}\cos(n\phi+\omega t) \tag{5}$$

$$v=V_0 e^{\frac{\alpha x}{a}}\sin(n\phi+\omega t) \tag{6}$$

$$w=W_0 e^{\frac{\alpha x}{a}}\cos(n\phi+\omega t) \tag{7}$$

into the equations of motion (1)-(3), yields:

$$\left(\left((-1+\mu)n^2+2\alpha^2\right)U_0+\left(V_0(1+\mu)n+2\mu W_0\right)\alpha\right)K-2U_0\left(-\rho h\omega^2a^2+n^2N_{\phi,i}\right)=0 \tag{8}$$

$$\begin{aligned}&-2\rho ha^4\left(\left(\omega^2+\Omega^2\right)V_0+2W_0\omega\Omega\right)+\\&+a^2\left(\left(2\left(K+N_{\phi,i}\right)n^2+K(\mu-1)\alpha^2+2N_{\phi,i}\right)V_0+n\left(KU_0(1+\mu)\alpha+2W_0\left(2N_{\phi,i}+K\right)\right)\right)+\\&+2D\left(\left(n^2+1/2\,\alpha^2(\mu-1)\right)V_0+W_0n^3-W_0\alpha^2n\right)=0\end{aligned} \tag{9}$$

$$\begin{aligned}&-\rho ha^4\left(\left(\omega^2+\Omega^2\right)W_0+2V_0\omega\Omega\right)+a^2\left(\left(n^2N_{\phi,i}+K+N_{\phi,i}\right)W_0+V_0\left(2N_{\phi,i}+K\right)n+KU_0\alpha\mu\right)+\\&+(n+\alpha)(n-\alpha)D\left(\left(-\alpha^2+n^2\right)W_0+nV_0\right)=0\end{aligned} \tag{10}$$

where $n$ is the number of circumferential waves, and $\omega$ is frequency. The three equations (8)-(10) can be condensed into a matrix form:

$$\begin{bmatrix}k_{1,1} & k_{1,2} & k_{1,3}\\ k_{2,1} & k_{2,2} & k_{2,3}\\ k_{3,1} & k_{3,2} & k_{3,3}\end{bmatrix}\begin{Bmatrix}U_0\\ V_0\\ W_0\end{Bmatrix}=0 \tag{11}$$

where:

$$\begin{aligned}k_{1,1}&=\left((\mu-1)n^2+2\alpha^2\right)K+2a^2\omega^2h\rho-2n^2N_{\phi,i}\\ k_{1,2}&=Kn(1+\mu)\alpha\\ k_{1,3}&=2K\mu\alpha\\ k_{2,1}&=Kn(1+\mu)\alpha a^2\\ k_{2,2}&=a^2\left(K(\mu-1)\alpha^2+2n^2K+2N_{\phi,i}+2n^2N_{\phi,i}\right)+D\left((\mu-1)\alpha^2+2n^2\right)-2\rho ha^4\left(\omega^2+\Omega^2\right)\end{aligned} \tag{12}$$

$$k_{2,3}=Dn^3+\left(\left(2N_{\phi,i}+K\right)a^2-D\alpha^2\right)n-2\omega\Omega\rho ha^4$$

$$k_{3,1}=K\mu\alpha a^2$$

$$k_{3,2}=Dn^3+\left(\left(2N_{\phi,i}+K\right)a^2-D\alpha^2\right)n-2\omega\Omega\rho ha^4$$

$$k_{3,3}=-\left(\omega^2+\Omega^2\right)\rho ha^4+\left(n^2N_{\phi,i}+K+N_{\phi,i}\right)a^2+D(n-\alpha)^2(n+\alpha)^2$$

The determinant of the matrix in Eq. (11) must vanish in order for the equations of motion to be satisfied. Assigning a value to the frequency $\omega$ and setting the determinant to zero yields a biquartic polynomial in $\alpha$:

$$a_8\alpha^8+a_6\alpha^6+a_4\alpha^4+a_2\alpha^2+a_0=0 \tag{13}$$

where the coefficients $a_r$ are given in the Appendix to this paper. Thus there are eight roots of the polynomial and so the radial component of the displacement can be expressed as:

$$w=W(x)\cos(n\phi+\omega t) \tag{14}$$

with:

$$W(x)=\sum_{r=1}^{8}B_r e^{\frac{\alpha_r x}{a}} \tag{15}$$

where $B_r$, with $r$=1…8, are eight generally complex constants. The roots of the polynomial (13) must be calculated numerically at this stage. As shown by Hu and Wah [4], for the usual range of parameters and $n\geq 1$ the roots can be expected to have the form $\pm\alpha_1, \pm i\gamma_2, \pm(p\pm iq)$ where $\alpha_1, \gamma_2, p, q$ are real and positive numbers. Then the radial component of the displacement is given by:

$$\begin{aligned}W(x)=&C_1\cosh\left(\frac{\alpha_1 x}{a}\right)+C_2\sinh\left(\frac{\alpha_1 x}{a}\right)+C_3\cos\left(\frac{\gamma_2 x}{a}\right)+C_4\sin\left(\frac{\gamma_2 x}{a}\right)+\\&+e^{\frac{px}{a}}\left(C_5\cos\left(\frac{qx}{a}\right)+C_6\sin\left(\frac{qx}{a}\right)\right)+e^{\frac{-px}{a}}\left(C_7\cos\left(\frac{qx}{a}\right)+C_8\sin\left(\frac{qx}{a}\right)\right)\end{aligned} \tag{16}$$

where $C_r$ are now real constants. From Eq. (8) - (10) and (12):

$$\left(\frac{U}{W}\right)_r=\frac{k_{1,2}k_{2,3}-k_{1,3}k_{2,2}}{k_{1,1}k_{2,2}-k_{2,1}k_{1,2}}; \tag{17}$$

and

$$\left(\frac{V}{W}\right)_r=\frac{k_{2,1}k_{1,3}-k_{2,3}k_{1,1}}{k_{1,1}k_{2,2}-k_{2,1}k_{1,2}} \tag{18}$$

Substituting for each root $\alpha_r$ into Eqs. (17) and (18), the expressions for $U(x)$ and $V(x)$ can be represented as:

$$\begin{aligned}V(x)=&d_1C_1\cosh\left(\frac{\alpha_1 x}{a}\right)+d_1C_2\sinh\left(\frac{\alpha_1 x}{a}\right)+d_3C_3\cos\left(\frac{\gamma_2 x}{a}\right)+d_3C_4\sin\left(\frac{\gamma_2 x}{a}\right)+\\&+e^{\frac{px}{a}}\left((d_5C_5+d_6C_6)\cos\left(\frac{qx}{a}\right)+(d_5C_6-d_6C_5)\sin\left(\frac{qx}{a}\right)\right)+e^{\frac{-px}{a}}\left((d_5C_7-d_6C_8)\cos\left(\frac{qx}{a}\right)+(d_5C_8+d_6C_7)\sin\left(\frac{qx}{a}\right)\right)\end{aligned} \tag{19}$$

$$\begin{aligned}U(x)=&d_2C_2\cosh\left(\frac{\alpha_1 x}{a}\right)+d_2C_1\sinh\left(\frac{\alpha_1 x}{a}\right)+d_4C_4\cos\left(\frac{\gamma_2 x}{a}\right)-d_4C_3\sin\left(\frac{\gamma_2 x}{a}\right)+\\&+e^{\frac{px}{a}}\left((d_7C_5+d_8C_6)\cos\left(\frac{qx}{a}\right)+(d_7C_6-d_8C_5)\sin\left(\frac{qx}{a}\right)\right)+e^{\frac{-px}{a}}\left((-d_7C_7+d_8C_8)\cos\left(\frac{qx}{a}\right)-(d_7C_8+d_8C_7)\sin\left(\frac{qx}{a}\right)\right)\end{aligned} \tag{20}$$

The constants $d_r$ can now be numerically calculated using Eqs. (17) and (18) as:

$$\begin{aligned}
d_1 &= (V/W)_r \ \text{ with } \alpha_r = \alpha_1 \\
d_2 &= (U/W)_r \ \text{ with } \alpha_r = \alpha_1 \\
d_3 &= (V/W)_r \ \text{ with } \alpha_r = \mathrm{i}\gamma_2 \\
d_4 &= \Im(U/W)_r \ \text{ with } \alpha_r = \mathrm{i}\gamma_2 \\
d_5 &= \Re(V/W)_r \ \text{ with } \alpha_r = p + \mathrm{i}q \\
d_6 &= \Im(V/W)_r \ \text{ with } \alpha_r = p + \mathrm{i}q \\
d_7 &= \Re(U/W)_r \ \text{ with } \alpha_r = p + \mathrm{i}q \\
d_8 &= \Im(U/W)_r \ \text{ with } \alpha_r = p + \mathrm{i}q
\end{aligned} \tag{21}$$

At each end of the shell marked with *x=const.* there are five resultant forces as shown in Figure 2, but the equations of motion are of maximum fourth order and can only accommodate for four boundary conditions. Thus the Kirchhoff effective shear stress resultant of the first kind, $V_{xz}$, and the Kirchhoff effective shear stress resultant of the second kind, $T_{x\phi}$, must be used that relate $Q_{x,z}$ to $M_{x,\phi}$, and $N_{x,\phi}$ to $M_{x,\phi}$, respectively [1].

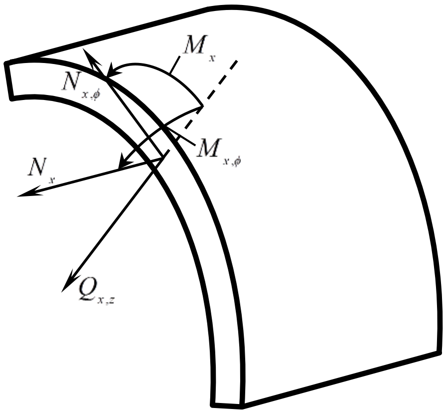


Figure 2: The boundary force resultants

These relations are [1]:

$$V_{xz} = Q_{xz} + \frac{1}{a}\frac{\partial M_{x\phi}}{\partial \phi} \tag{22}$$

and

$$T_{x\phi} = N_{x\phi} + \frac{1}{a} M_{x\phi} \tag{23}$$

The four boundary conditions for a shell with both ends free are:

$$N_x = 0; \ \text{ i.e. } \ a\frac{\partial u}{\partial x} + \mu\, w + \mu \frac{\partial v}{\partial \phi} = 0 \tag{24}$$

$$M_x = 0; \ \text{ i.e. } \ a^2 \frac{\partial^2 w}{\partial x^2} + \mu\left(\frac{\partial^2 w}{\partial \phi^2} - \frac{\partial v}{\partial \phi}\right) = 0 \tag{25}$$

$$V_{xz} = 0; \ \text{ i.e. } \ (2-\mu)\frac{\partial^3 w}{\partial x \partial \phi^2} - \frac{\partial^2 v}{\partial x \partial \phi} + a^2 \frac{\partial^3 w}{\partial x^3} = 0 \tag{26}$$

$$T_{x\phi} = 0; \ \text{ i.e. } \ -2h^2 \frac{\partial^2 w}{\partial x \partial \phi} + \left(12a^2 + h^2\right)\frac{\partial v}{\partial x} + 12a\frac{\partial u}{\partial \phi} = 0 \tag{27}$$

These can be separately satisfied for symmetric and anti-symmetric modes by assuming the origin in the middle section of the cylinder and the boundaries at $x=\pm L/2$. Here the radial and the axial displacement components share the symmetry or anti-symmetry properties and the circumferential displacement component is opposite to the radial and axial displacement components in terms of symmetry or anti-symmetry. In case the modes are symmetric, the radial component of the displacement can be shortened by recognising that $e^{\pm px/a}=\cosh(px/a)\pm\sinh(px/a)$ and ensuring that the anti-symmetric terms in Eq. (16) vanish. Then it must be $C_2=C_4=0,\ C_5=C_7$, and $C_6=C_8$, so that Eqs. (16), (19) and (20) reduce to:

$$W(x)=C_1\cosh\left(\frac{\alpha_1 x}{a}\right)+C_3\cos\left(\frac{\gamma_2 x}{a}\right)+F_1\cosh\left(\frac{px}{a}\right)\cos\left(\frac{qx}{a}\right)+F_2\sinh\left(\frac{px}{a}\right)\sin\left(\frac{qx}{a}\right) \tag{28}$$

$$V(x)=d_1C_1\cosh\left(\frac{\alpha_1 x}{a}\right)+d_3C_3\cos\left(\frac{\gamma_2 x}{a}\right)+\left(d_5F_1+d_6F_2\right)\cosh\left(\frac{px}{a}\right)\cos\left(\frac{qx}{a}\right)+\left(d_5F_2-d_6F_1\right)\sinh\left(\frac{px}{a}\right)\sin\left(\frac{qx}{a}\right) \tag{29}$$

$$U(x)=d_2C_1\sinh\left(\frac{\alpha_1 x}{a}\right)-d_4C_3\sin\left(\frac{\gamma_2 x}{a}\right)+\left(d_7F_1+d_8F_2\right)\sinh\left(\frac{px}{a}\right)\cos\left(\frac{qx}{a}\right)+\left(d_7F_2-d_8F_1\right)\cosh\left(\frac{px}{a}\right)\sin\left(\frac{qx}{a}\right) \tag{30}$$

Substituting Eq. (28) into (14) for the radial displacement component, and also Eqs. (29) and (30) into analogue expressions for tangential and axial displacement components gives a set of displacement components *u*,*v* and *w*. These can be substituted into Eqs. (24)-(27), which after assuming $x=L/2$ and putting into a matrix form gives:

$$\begin{bmatrix}
t_{1,1}\cosh\left(\frac{\alpha_1 L}{2a}\right) & t_{1,2}\cos\left(\frac{\gamma_2 L}{2a}\right) & \begin{matrix} t_{1,3,\mathrm{A}}\cos\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{1,3,\mathrm{B}}\sin\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} & \begin{matrix} t_{1,4,\mathrm{A}}\cos\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{1,4,\mathrm{B}}\sin\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} \\
t_{2,1}\cosh\left(\frac{\alpha_1 L}{2a}\right) & t_{2,2}\cos\left(\frac{\gamma_2 L}{2a}\right) & \begin{matrix} t_{2,3,\mathrm{A}}\cos\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{a}\right)+ \\ +t_{2,3,\mathrm{B}}\sin\left(\frac{qL}{a}\right)\sinh\left(\frac{pL}{a}\right)\end{matrix} & \begin{matrix} t_{2,4,\mathrm{A}}\cos\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{2,4,\mathrm{B}}\sin\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} \\
t_{3,1}\sinh\left(\frac{\alpha_1 L}{2a}\right) & t_{3,2}\sin\left(\frac{\gamma_2 L}{2a}\right) & \begin{matrix} t_{3,3,\mathrm{A}}\sin\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{3,3,\mathrm{B}}\cos\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} & \begin{matrix} t_{3,4,\mathrm{A}}\sin\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{3,4,\mathrm{B}}\cos\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} \\
t_{4,1}\sinh\left(\frac{\alpha_1 L}{2a}\right) & t_{4,2}\sin\left(\frac{\gamma_2 L}{2a}\right) & \begin{matrix} t_{4,3,\mathrm{A}}\sin\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{4,3,\mathrm{B}}\cos\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix} & \begin{matrix} t_{4,4,\mathrm{A}}\sin\left(\frac{qL}{2a}\right)\cosh\left(\frac{pL}{2a}\right)+ \\ +t_{4,4,\mathrm{B}}\cos\left(\frac{qL}{2a}\right)\sinh\left(\frac{pL}{2a}\right)\end{matrix}
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_3 \\ F_1 \\ F_2 \end{Bmatrix}=0 \tag{31}$$

The coefficients $t_{r,s,\sim}$ depend on the roots $\alpha_1,\gamma_2,p,q$, the constants $d_r$, $n$ and $\mu$, and are listed in the Appendix to the paper. In order for the boundary conditions to be satisfied, the determinant of the matrix in Eq. (31) must vanish. The determinant, after being divided by $\sinh\left(\frac{pL}{2a}\right)\cdot\sinh\left(\frac{\alpha_1 L}{2a}\right)\cdot\cosh\left(\frac{pL}{2a}\right)$ for better numerical behaviour, is given by:

$$\begin{aligned}
&\left(\left(\tanh(\theta_3)b_1b_2\cos(\theta_2)+\sin(\theta_2)b_3\right)\coth(\theta_1)+b_4\cos(\theta_2)-\sin(\theta_2)\coth(\theta_3)b_5b_6\right)\cos(\theta_4)^2+\\
&+\left(\left(b_7b_2\cos(\theta_2)+\left(b_9\coth(\theta_3)+\tanh(\theta_3)b_8\right)\sin(\theta_2)\right)\coth(\theta_1)+\left(b_{11}\coth(\theta_3)+b_{10}\tanh(\theta_3)\right)\cos(\theta_2)+\sin(\theta_2)b_5b_{12}\right)\times\\
&\times\sin(\theta_4)\cos(\theta_4)+\left(\left(\coth(\theta_3)b_{13}b_2\cos(\theta_2)+b_{14}\sin(\theta_2)\right)\coth(\theta_1)+b_{15}\cos(\theta_2)+\sin(\theta_2)\tanh(\theta_3)b_5b_{16}\right)\sin(\theta_4)^2=0
\end{aligned} \tag{32}$$

where $\theta_1=\frac{\alpha_1 L}{2a}$, $\theta_2=\frac{\gamma_2 L}{2a}$, $\theta_3=\frac{pL}{2a}$ and $\theta_4=\frac{qL}{2a}$, with coefficients $b_r$ given in the Appendix to the paper. The zeroes of the left hand side of Eq.(32) in fact represent the length of the shell whose $m,n$ mode resonates at the assumed frequency $\omega_{m,n}$, where the lowest zero (the smallest length) is for $m$=1 symmetric mode, the next one is for $m$=3 symmetric mode etc. In case of the non-rotating shell ($\Omega=0$), assuming frequencies $\pm|\omega_{m,n}|$ yields the same length for either the positive or the negative frequency $\omega_{m,n}$, however for $\Omega\neq 0$ this is not the case, since the forward and backward travelling waves are now characterised by different speeds and thus different resonance frequencies.

In case of anti-symmetric modes it must be $C_1=C_3=0$, $C_5=-C_7$, and $C_6=C_8$ so that:

$$W(x)=C_2\sinh\left(\frac{\alpha_1 x}{a}\right)+C_4\sin\left(\frac{\gamma_2 x}{a}\right)+F_3\sinh\left(\frac{px}{a}\right)\cos\left(\frac{qx}{a}\right)+F_4\cosh\left(\frac{px}{a}\right)\sin\left(\frac{qx}{a}\right) \tag{33}$$

with similar expressions for $V(x)$ and $U(x)$. After recalculating the determinant for the anti-symmetric modes an expression analogue to Eq. (32) results, which can be obtained by substituting $\tanh(\theta_1)\rightleftarrows\coth(\theta_1)$, $\tanh(\theta_3)\rightleftarrows\coth(\theta_3)$, $\cos(\theta_2)\rightarrow\sin(\theta_2)$, and $\sin(\theta_2)\rightarrow-\cos(\theta_2)$ into Eq. (32). Then the zeroes of the expression represent the length of the shell whose $m,n$ mode resonates at the assumed frequency $\omega_{m,n}$, where the lowest zero is for $m$=0 anti-symmetric mode (Love mode), the next one is for $m$=2 anti-symmetric mode etc. The symmetric Rayleigh type modes ($m$=-1) do require a separate treatment which is not covered within this study. In order to calculate resonance frequencies for a shell with given length it is necessary to iterate until the length resulting from finding a zero of the determinant matches the real length of the shell to a desired precision.

## RESULTS

Natural frequencies of an example rotating shell are calculated next and plotted against the circumferential wavenumber n in Figure 3 for a non-rotating shell and for shell rotating at increasing speeds. The physical and geometrical properties of the shell are as follows: $a$=0.1m, $h$=0.002m, $L$=0.2m, $\rho$=1450kg/m$^3$ $\mu$=0.45 and $E$=0.45 GPa. The left hand side branches (negative $n$) show the absolute values of the natural frequencies for the forward travelling waves, and the right hand side branches (positive $n$) are for the backward travelling waves. Each branch is for $m$=const. and only the bending waves are considered which have the lowest natural frequencies and which are of the foremost practical interest. As can be seen in Figure 3 the rotation causes asymmetries in the plots as the forward travelling waves initially decrease their natural frequencies for small rotation speeds and low order modes (i.e. $m$=1,$n$=2), however for higher rotation speeds and higher mode orders the tendency is that the natural frequencies of both the forward travelling and backward travelling waves increase with the rotation speed. This is predominantly due to the tension caused by the centrifugal forces in the belt which increase with squared rotation speed.

## CONCLUSIONS

Free vibrations of rotating thin cylindrical shells are considered. A mathematical model is developed to calculate mode shapes and natural frequencies of a rotating shell having free ends. The boundary conditions are satisfied exactly. The proposed method requires an assumption of a natural frequency and then the calculation of the length of the rotating shell that can vibrate at that natural frequency. A number of natural frequencies have been calculated by iteration until the prescribed length is matched as a function of the rotation speed for the example shell. The behaviour of the natural frequencies is such that in principle they increase with the increase in rotation speed, except for some low order modes at low rotation speeds. This is predominantly due to a stiffening effect caused by the centrifugal forces.

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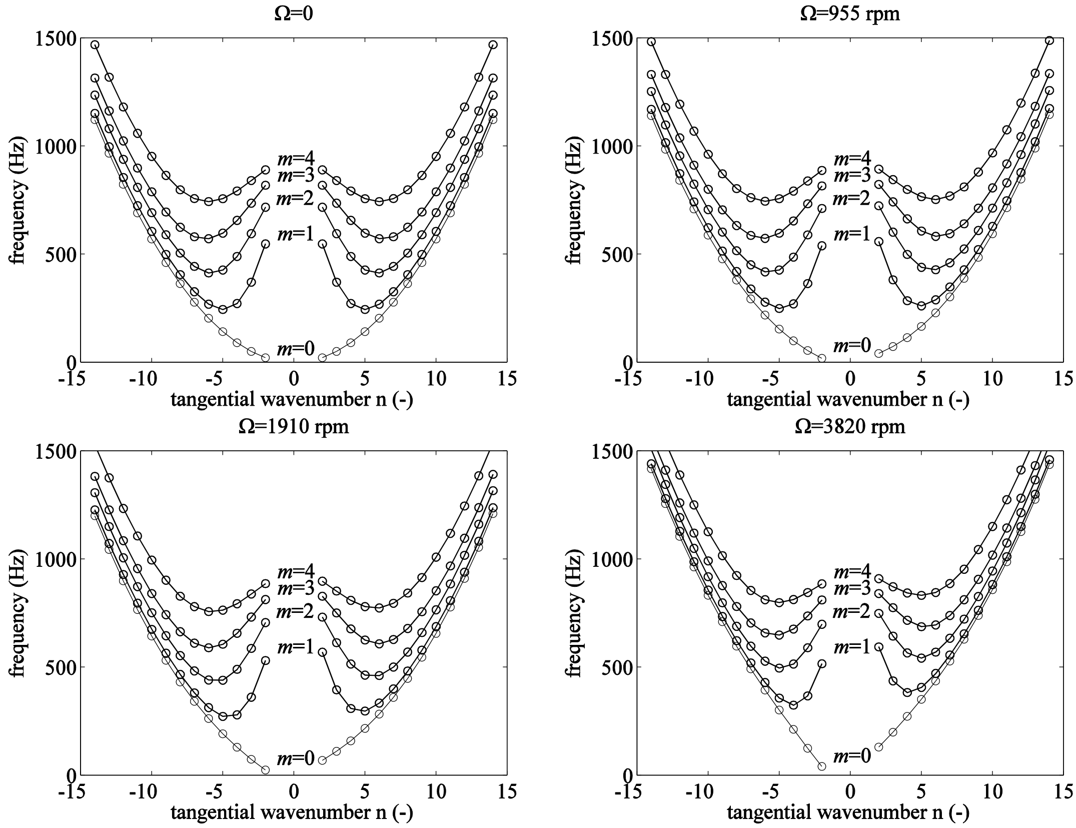


Figure 3: Natural frequencies of the shell with free boundaries for a non-rotating shell and for rotating shells with three different rotation speeds.

APPENDIX

The coefficients $a_i$ are:

$$a_0=2a^2\left(\left(K(-1+\mu)-2N_{\phi,i}\right)n^2+2a^2\omega^2 h\rho\right)\times$$
$$\left(\begin{array}{l} D\left(N_{\phi,i}+K\right)n^6+\left(\left(-hDD\left(\omega^2+\Omega^2\right)\rho+N_{\phi,i}\left(N_{\phi,i}+K\right)\right)a^2-2D\left(N_{\phi,i}+K\right)\right)n^4+ \\ +4n^3D\rho h\omega\Omega a^2+\left(-\rho h\left(\omega^2+\Omega^2\right)\left(2N_{\phi,i}+K\right)a^4+\left(-hD\left(\omega^2+\Omega^2\right)\rho-2N_{\phi,i}\left(N_{\phi,i}+K\right)\right)a^2+D\left(N_{\phi,i}+K\right)\right)n^2+ \\ +4a^4h\rho\Omega\omega\left(2N_{\phi,i}+K\right)n+\left(h^2\rho^2(\Omega-\omega)^2(\Omega+\omega)^2a^4-\rho h\left(\omega^2+\Omega^2\right)\left(2N_{\phi,i}+K\right)a^2+N_{\phi,i}\left(N_{\phi,i}+K\right)\right)a^2 \end{array}\right)$$

$a_2=a_{2,1}+a_{2,2}+a_{2,3}$ with

$$a_{2,1}=4\rho ha^6\left(\begin{array}{l} \left(\left(n^2+\mu+3/2\right)\omega^2-2n\Omega(\mu+2)\omega+\Omega^2\left(\mu+n^2+1\right)\right)(-1+\mu)K^2+ \\ +\left(\begin{array}{l} -1/2\,a^2h\rho(\mu-3)\omega^4+\left(-1/2\,a^2h\rho(3+\mu)\Omega^2+N_{\phi,i}\left((\mu-3)n^2+1/2\,\mu-5/2\right)\right)\omega^2+ \\ +8\Omega\omega nN_{\phi,i}+1/2\,\Omega^2\left(2h\rho a^2\Omega^2+N_{\phi,i}\left(-4+(\mu-5)n^2\right)\right) \end{array}\right)K+ \\ +2\left(\left(n^2+1/4-1/4\,\mu\right)\omega^2-2\Omega\omega n+\Omega^2\left(n^2+1/4-1/4\,\mu\right)\right)\rho Dh\omega^2 \end{array}\right)$$

$$a_{2,2}=-4a^4\left(\begin{array}{l}\left(N_{\phi,i}(-1+\mu)K^2+\left(-h\rho\left(\left(3/2\,\omega^2+\Omega^2\right)\mu-9/2\,\omega^2-2\,\Omega^2\right)D+1/2\,N_{\phi,i}{}^2(\mu-3)\right)K+2\,N_{\phi,i}hD\,\rho\left(\Omega^2+2\,\omega^2\right)\right)n^4+\\ +2\,\Omega\,\omega\,D\,h\left((\mu-3)K-2\,N_{\phi,i}\right)\rho\,n^3+n\left(\begin{array}{l}-(\mu+3/2)(-1+\mu)N_{\phi,i}K^2+\\ +\left(1/4\left(\left(\omega^2+\Omega^2\right)\mu^2+\left(-2\,\Omega^2-2\,\omega^2\right)\mu-3\,\omega^2+5\Omega^2\right)h\rho D+1/2\,N_{\phi,i}{}^2(3+\mu)\right)K-\\ -1/2\left(\left(\Omega^2+2\,\omega^2\right)\mu+2\,\omega^2-\Omega^2\right)D\,hN_{\phi,i}\rho\end{array}\right)^2+\\ +N_{\phi,i}\left(\mu^2-1\right)K^2+\left(-1/2\,\omega^2h\rho\,(-1+\mu)D-N_{\phi,i}{}^2\right)K-1/2\,N_{\phi,i}hD\,\rho\,\omega^2(-1+\mu)\end{array}\right)$$

$$a_{2,3}=-8D\,n^2\left(\begin{array}{l}a^2\left(\begin{array}{l}\left((\mu-1)K^2+3/4\,N_{\phi,i}(\mu-3)K-N_{\phi,i}{}^2\right)n^4+\\ +\left(\left(-1/2\,\mu^2+3/2-\mu\right)K^2-1/8\,N_{\phi,i}\left(-19+2\,\mu+\mu^2\right)K+\left(-1/4\,\omega^2h\rho D+1/4\,N_{\phi,i}{}^2\right)\mu+1/4\,\omega^2h\rho D+3/4\,N_{\phi,i}{}^2\right)n^2+\\ +\left(3/8\,\mu^2-5/8+1/4\,\mu\right)K^2-1/8\,N_{\phi,i}\left(\mu^2-4\,\mu+7\right)K+1/4\,N_{\phi,i}{}^2(\mu-1)\end{array}\right)-\\ -1/8\,D\left(K(-1+\mu)-2\,N_{\phi,i}\right)n^4(\mu-1)\end{array}\right)$$

$$\begin{array}{l}a_4=-2\left(\omega^2+\Omega^2\right)\left((\mu-1)K^2+2\,\omega^2h\rho D\right)h\rho\,a^6+\\ +a^4\left(\begin{array}{l}-2\,(1+\mu)(-1+\mu)^2K^3+2\,N_{\phi,i}\left(1+n^2\right)(\mu-1)K^2-\\ -2\,D\,h\rho\left(\left(\left(\Omega^2+3\,\omega^2\right)\mu-9\,\omega^2-5\,\Omega^2\right)n^2+8\,\Omega\,\omega\,n+\left(\omega^2+\Omega^2\right)(\mu-1)\right)K+4\left(\left(\Omega^2+2\,\omega^2\right)n^2+\omega^2\right)D\,hN_{\phi,i}\rho\end{array}\right)+\\ +12Da^2\left(\begin{array}{l}\left(n^4+(-1/3\,\mu-2/3)n^2-1/6\,\mu^2+1/6\right)(\mu-1)K^2+\\ +1/2\,N_{\phi,i}\left((\mu-3)n^4+(2/3+2/3\,\mu)n^2+1/3\,\mu-1/3\right)K-1/3\,n^2\left(h\rho\,\omega^2(\mu-1)D+N_{\phi,i}{}^2\left(1+n^2\right)\right)\end{array}\right)-2\left((\mu-2)K-2\,N_{\phi,i}\right)D^2(\mu-1)n^4\end{array}$$

$$a_6=-8\,D\left(\begin{array}{l}n^2a^2(\mu-1)K^2+\\ +\left(-1/4\,h\rho\left(-3\,\omega^2+\omega^2\mu-2\,\Omega^2\right)a^4+1/4\left(-2+(\mu-3)n^2\right)N_{\phi,i}a^2-1/8\,n^2D\,(\mu-1)(\mu-5)\right)K+\\ +1/4\,D\,(\mu-1)\left(-a^2\omega^2h\rho+n^2N_{\phi,i}\right)\end{array}\right)$$

$$a_8=2\,KD\,(\mu-1)\left(D+a^2K\right)$$

The coefficients $t_{r,s,\sim}$ are:

$$t_{1,1}=\left(\mu+\mu\,nd_1+d_2\alpha_1\right)$$
$$t_{2,1}=-\left(n(n+d_1)\mu-\alpha_1{}^2\right)$$
$$t_{3,1}=\left((\mu-2)n^2-nd_1+\alpha_1{}^2\right)\alpha_1$$
$$t_{4,1}=-\left(-12\,a^2d_1\alpha_1+12\,a^2d_2n-h^2\alpha_1d_1-2\,h^2\alpha_1n\right)$$
$$t_{1,2}=\left(-\gamma_2d_4+\mu\,nd_3+\mu\right)$$
$$t_{2,2}=-\left(n(n+d_3)\mu+\gamma_2{}^2\right)$$
$$t_{3,2}=-\left((\mu-2)n^2-nd_3-\gamma_2{}^2\right)\gamma_2$$
$$t_{4,2}=\left(-12\,a^2d_3\gamma_2+12\,a^2d_4n-h^2\gamma_2d_3-2\,h^2\gamma_2n\right)$$
$$t_{1,3,A}=-\left((-1-nd_5)\mu-pd_7+qd_8\right)$$
$$t_{1,3,B}=-\left(nd_6\mu+pd_8+qd_7\right)$$
$$t_{2,3,A}=-\left(n(n+d_5)\mu-p^2+q^2\right)$$
$$t_{2,3,B}=\left(nd_6\mu-2\,pq\right)$$

$$t_{3,3,A} = -\left((\mu-2)qn^2 + \left(-pd_6 - qd_5\right)n - q^3 + 3p^2q\right)$$
$$t_{3,3,B} = \left((\mu-2)pn^2 + \left(-pd_5 + qd_6\right)n - 3pq^2 + p^3\right)$$
$$t_{4,3,A} = 12\left(\left(nd_8 - qd_5 - pd_6\right)a^2 - 1/6\,h^2\left(1/2\,pd_6 + \left(n + 1/2\,d_5\right)q\right)\right)$$
$$t_{4,3,B} = -\left(\left(nd_7 + qd_6 - pd_5\right)a^2 - 1/6\left(\left(n + 1/2\,d_5\right)p - 1/2\,qd_6\right)h^2\right)12$$
$$t_{1,4,A} = \left(nd_6\mu + pd_8 + qd_7\right)$$
$$t_{1,4,B} = -\left(\left(-1 - nd_5\right)\mu - pd_7 + qd_8\right)$$
$$t_{2,4,A} = -\left(nd_6\mu - 2pq\right)$$
$$t_{2,4,B} = -\left(n\left(n + d_5\right)\mu - p^2 + q^2\right)$$
$$t_{3,4,A} = \left((\mu-2)pn^2 + \left(-pd_5 + qd_6\right)n - 3pq^2 + p^3\right)$$
$$t_{3,4,B} = \left((\mu-2)qn^2 + \left(-pd_6 - qd_5\right)n - q^3 + 3p^2q\right)$$
$$t_{4,4,A} = -12\left(\left(nd_7 + qd_6 - pd_5\right)a^2 - 1/6\left(\left(n + 1/2\,d_5\right)p - 1/2\,qd_6\right)h^2\right)$$
$$t_{4,4,B} = -12\left(\left(nd_8 - qd_5 - pd_6\right)a^2 - 1/6\,h^2\left(1/2\,pd_6 + \left(n + 1/2\,d_5\right)q\right)\right)$$

The coefficients $b_r$ are:

$$b_1 = t_{3,3,B}t_{4,4,B} - t_{3,4,B}t_{4,3,B}$$
$$b_2 = -t_{2,1}t_{1,2} + t_{1,1}t_{2,2}$$
$$b_3 = \left(\left(-t_{2,3,A}t_{4,4,B} + t_{4,3,B}t_{2,4,A}\right)t_{3,2} + t_{4,2}\left(t_{2,3,A}t_{3,4,B} - t_{2,4,A}t_{3,3,B}\right)\right)t_{1,1} + t_{2,1}\left(\left(-t_{4,3,B}t_{1,4,A} + t_{1,3,A}t_{4,4,B}\right)t_{3,2} + t_{4,2}\left(-t_{3,4,B}t_{1,3,A} + t_{3,3,B}t_{1,4,A}\right)\right)$$
$$b_4 = \left(\left(t_{2,3,A}t_{4,4,B} - t_{4,3,B}t_{2,4,A}\right)t_{3,1} - t_{4,1}\left(t_{2,3,A}t_{3,4,B} - t_{2,4,A}t_{3,3,B}\right)\right)t_{1,2} - \left(\left(-t_{4,3,B}t_{1,4,A} + t_{1,3,A}t_{4,4,B}\right)t_{3,1} + t_{4,1}\left(-t_{3,4,B}t_{1,3,A} + t_{3,3,B}t_{1,4,A}\right)\right)t_{2,2}$$
$$b_5 = -t_{4,1}t_{3,2} + t_{3,1}t_{4,2}$$
$$b_6 = -t_{2,4,A}t_{1,3,A} + t_{2,3,A}t_{1,4,A}$$
$$b_7 = -t_{3,4,A}t_{4,3,B} - t_{3,4,B}t_{4,3,A} + t_{3,3,A}t_{4,4,B} + t_{3,3,B}t_{4,4,A}$$
$$b_8 = \left(\left(t_{4,3,B}t_{2,4,B} - t_{2,3,B}t_{4,4,B}\right)t_{3,2} + t_{4,2}\left(t_{2,3,B}t_{3,4,B} - t_{2,4,B}t_{3,3,B}\right)\right)t_{1,1} - t_{2,1}\left(\left(t_{4,3,B}t_{1,4,B} - t_{1,3,B}t_{4,4,B}\right)t_{3,2} + t_{4,2}\left(-t_{1,4,B}t_{3,3,B} + t_{1,3,B}t_{3,4,B}\right)\right)$$
$$b_9 = \left(\left(t_{4,3,A}t_{2,4,A} - t_{2,3,A}t_{4,4,A}\right)t_{3,2} + t_{4,2}\left(t_{2,3,A}t_{3,4,A} - t_{2,4,A}t_{3,3,A}\right)\right)t_{1,1} + t_{2,1}\left(\left(-t_{4,3,A}t_{1,4,A} + t_{1,3,A}t_{4,4,A}\right)t_{3,2} + t_{4,2}\left(-t_{3,4,A}t_{1,3,A} + t_{3,3,A}t_{1,4,A}\right)\right)$$
$$b_{10} = \left(\left(-t_{4,3,B}t_{2,4,B} + t_{2,3,B}t_{4,4,B}\right)t_{3,1} - t_{4,1}\left(t_{2,3,B}t_{3,4,B} - t_{2,4,B}t_{3,3,B}\right)\right)t_{1,2} + t_{2,2}\left(\left(t_{4,3,B}t_{1,4,B} - t_{1,3,B}t_{4,4,B}\right)t_{3,1} + t_{4,1}\left(-t_{1,4,B}t_{3,3,B} + t_{1,3,B}t_{3,4,B}\right)\right)$$
$$b_{11} = \left(\left(t_{2,3,A}t_{4,4,A} - t_{4,3,A}t_{2,4,A}\right)t_{3,1} - t_{4,1}\left(t_{2,3,A}t_{3,4,A} - t_{2,4,A}t_{3,3,A}\right)\right)t_{1,2} - t_{2,2}\left(\left(-t_{4,3,A}t_{1,4,A} + t_{1,3,A}t_{4,4,A}\right)t_{3,1} + t_{4,1}\left(-t_{3,4,A}t_{1,3,A} + t_{3,3,A}t_{1,4,A}\right)\right)$$
$$b_{12} = -t_{1,4,B}t_{2,3,A} - t_{1,4,A}t_{2,3,B} + t_{2,4,B}t_{1,3,A} + t_{1,3,B}t_{2,4,A}$$
$$b_{13} = -t_{3,4,A}t_{4,3,A} + t_{3,3,A}t_{4,4,A}$$
$$b_{14} = \left(\left(-t_{2,3,B}t_{4,4,A} + t_{4,3,A}t_{2,4,B}\right)t_{3,2} + t_{4,2}\left(t_{2,3,B}t_{3,4,A} - t_{2,4,B}t_{3,3,A}\right)\right)t_{1,1} - t_{2,1}\left(\left(t_{4,3,A}t_{1,4,B} - t_{1,3,B}t_{4,4,A}\right)t_{3,2} + t_{4,2}\left(t_{1,3,B}t_{3,4,A} - t_{1,4,B}t_{3,3,A}\right)\right)$$
$$b_{15} = \left(\left(t_{2,3,B}t_{4,4,A} - t_{4,3,A}t_{2,4,B}\right)t_{3,1} - t_{4,1}\left(t_{2,3,B}t_{3,4,A} - t_{2,4,B}t_{3,3,A}\right)\right)t_{1,2} + \left(\left(t_{4,3,A}t_{1,4,B} - t_{1,3,B}t_{4,4,A}\right)t_{3,1} + t_{4,1}\left(t_{1,3,B}t_{3,4,A} - t_{1,4,B}t_{3,3,A}\right)\right)t_{2,2}$$
$$b_{16} = t_{1,3,B}t_{2,4,B} - t_{1,4,B}t_{2,3,B}$$

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